Threshold Policies with Tight Guarantees for Online Selection with Convex Costs

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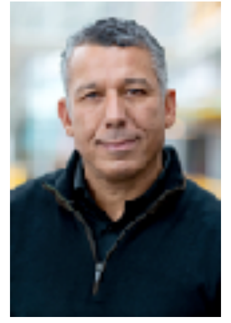
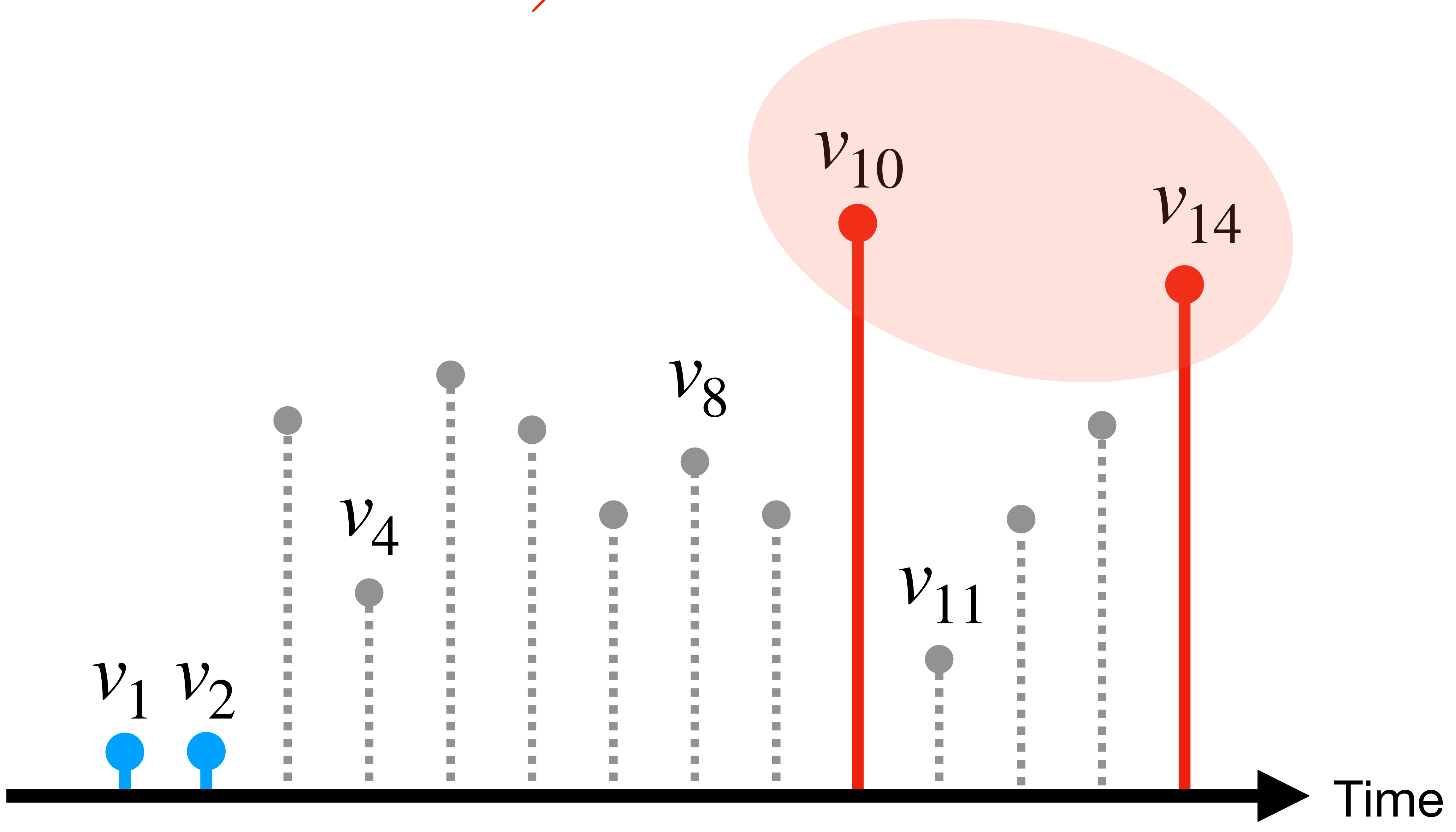
Joint work with Siyuan Yu, Raouf Boutaba, and Alberto Leon-Garcia



WINE 2023 - The 19th Conference on Web and Internet Economics, Shanghai, China

- **Selling** k items for a sequence of buyers
- Each buyer $t = 1,2,\cdots,T$ makes an **offer** v_t
- **Accept it and sell 1 item**, or wait for next one
- Once sold, cannot be reclaimed

$$\alpha = \max_{\sigma} \frac{\mathrm{Opt}(\sigma)}{\mathbb{E}[\mathrm{Alg}(\sigma)]}$$



e.g., $k = 2,\ T = 14$

Online Selection Problems

- A set of T items (one at a time)
- Select a subset $S \subset T$ of items (with possible constraints)
- Maximize objective $v(S)$

WINE 2008

Budget Constrained Bidding in Keyword Auctions and Online Knapsack Problems

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Online Selection Problems against Constrained Adversary

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ICML 2021

Pareto-Optimal Learning-Augmented Algorithms for Online Conversion Problems

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NeurIPS 2021

Optimal Search and One-Way Trading Online Algorithms

R. El-Yaniv,[1] A. Fiat,[2] R. M. Karp,[3] and G. Turpin[4]

Algorithmica 2001

Optimal Algorithms for k-Search with Application in Option Pricing

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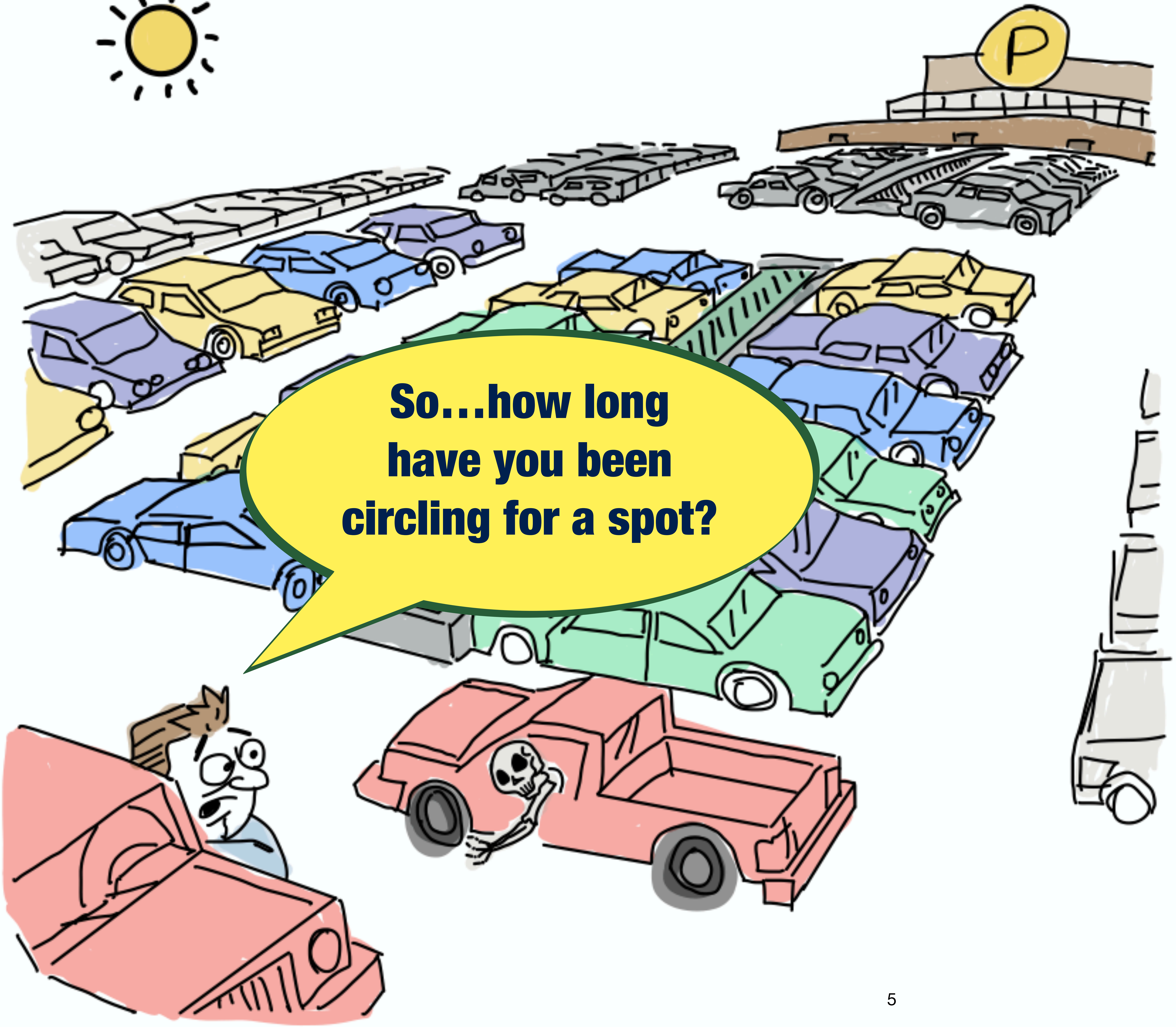
ESA 2007

OSCC: Online Selection w/ Convex Costs

- A set of T items (one at a time)
- Select a subset $S \subset T$ of items (with possible constraints)
- Maximize objective $v(S) - f(S)$

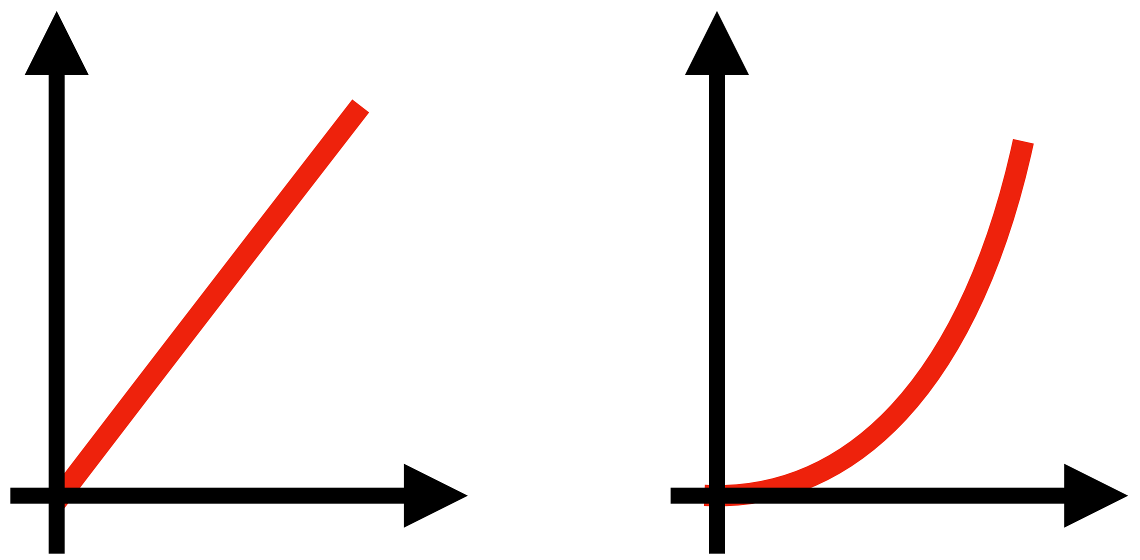
convex & monotone

$$f(y) = \begin{cases} \text{convex \& monotone} & \text{if } y \in [0,15], \\ +\infty & \text{otherwise}. \end{cases}$$


So...how long
have you been
circling for a spot?


$ 6
$ 3
$ 2.5
5 kg
10 kg
1 kg
15 kg

- A set of T items (one at a time)
- Select a subset $S \subset T$ of items (with possible constraints)
- Maximize objective $v(S) - f(S)$



$$\max_{x_t \in \mathbb{X}_t} \sum_t \sum_b v_t^b x_t^b - \sum_j f_j\Big(\sum_t \sum_b w_j^b x_t^b\Big)$$

convex & monotone production costs

$$\left| \begin{array}{l} \sum_b x_t^b \le 1, \forall t \\ x_t^b = \{0,1\}, \forall t, b \end{array} \right.$$

At most **one bundle** is selected

FOCS 2011

Welfare and Profit Maximization with Production Costs

Avrim Blum* Anupam Gupta* Yishay Mansour† Ankit Sharma*

Welfare Maximization with Production Costs: A Primal Dual Approach

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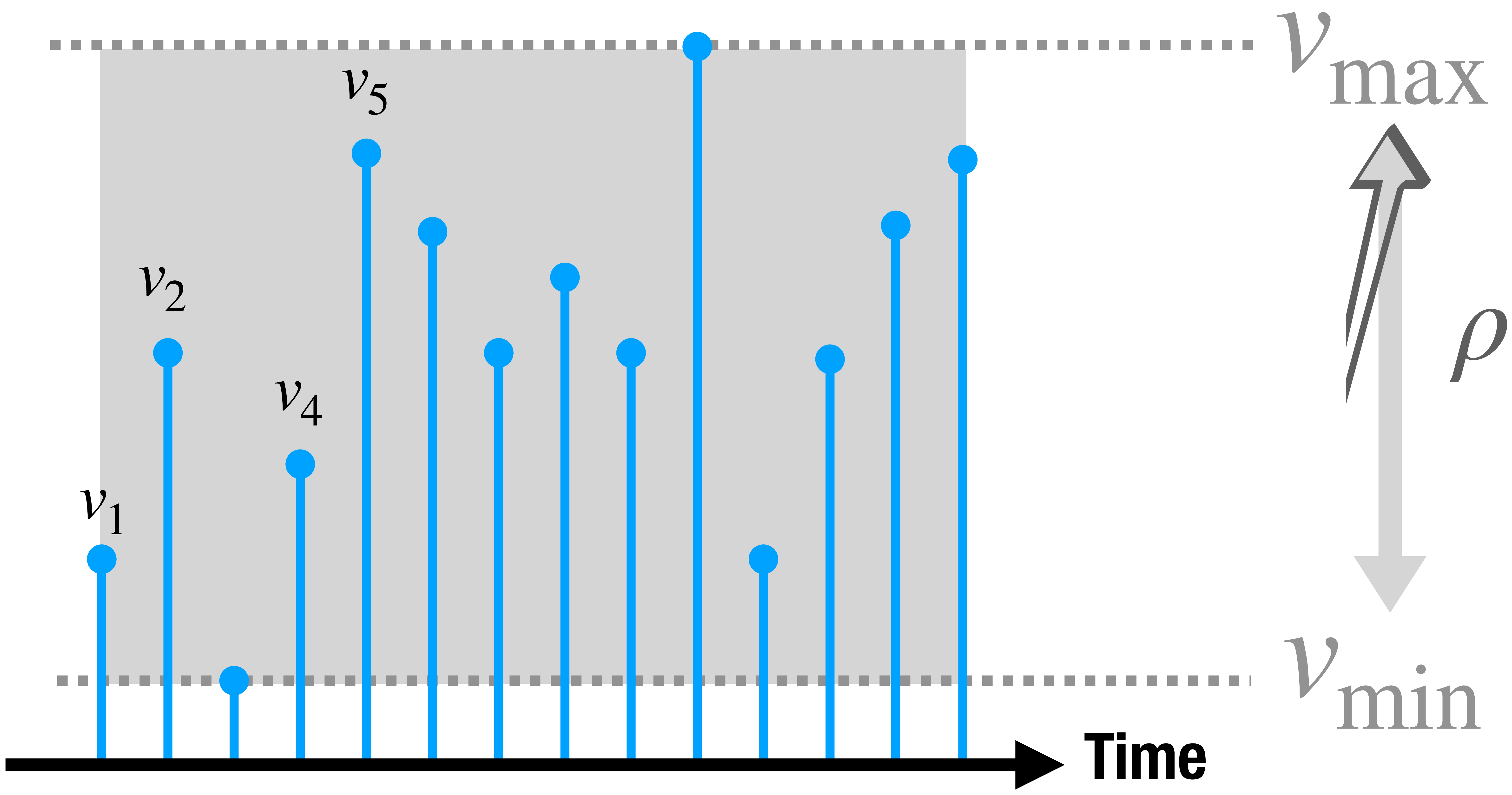
SODA 2015

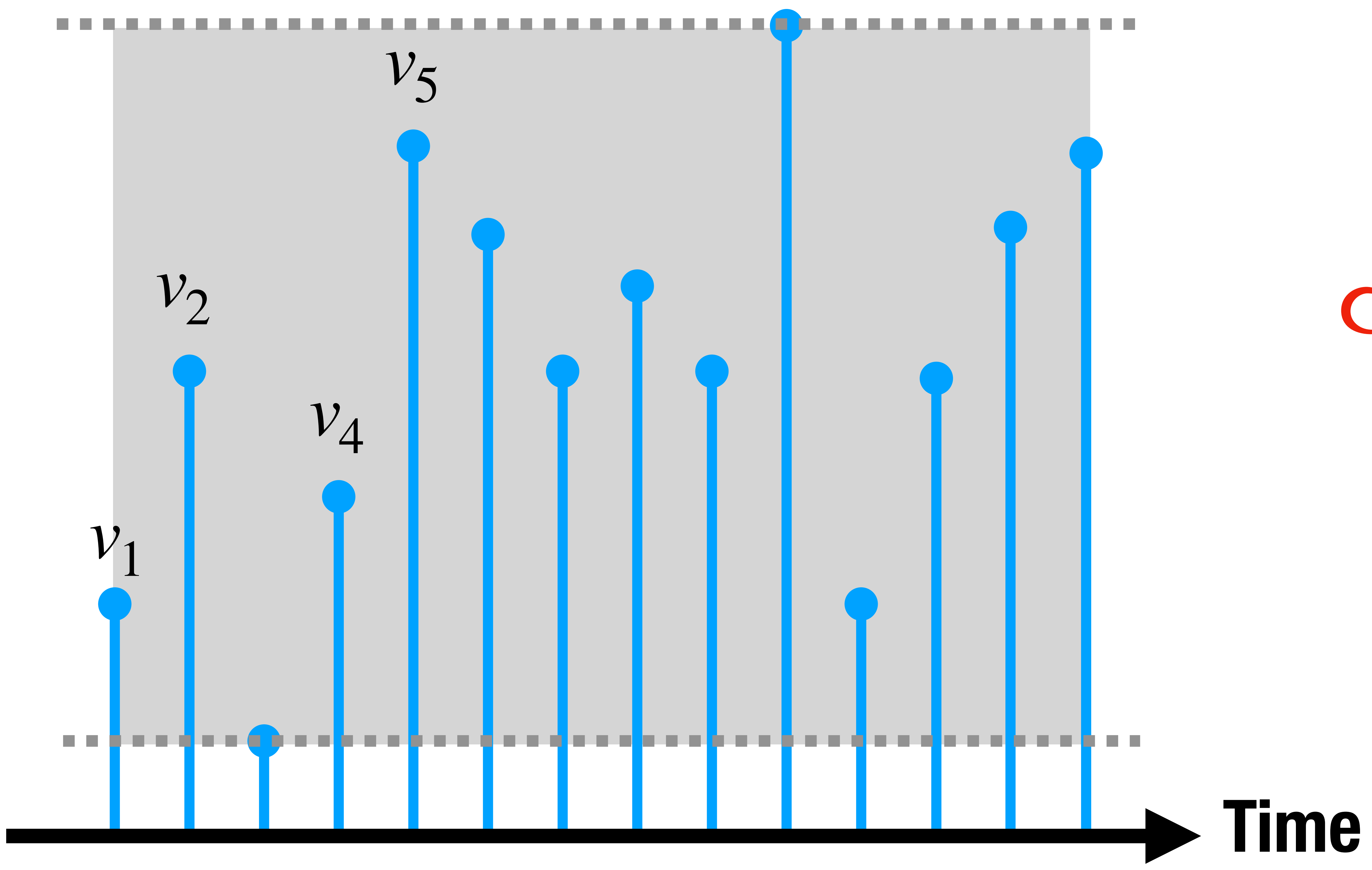
OSCC: Basic Setting



$$\rho \triangleq v_{\max}/v_{\min}$$

$$\begin{aligned} \underset{\{x_t\}_{\forall t}}{\text{maximize}} \quad & \sum_{t=1}^{T} v_t x_t - f\left(\sum_{t=1}^{T} x_t\right) \\ \text{subject to} \quad & \sum_{t=1}^{T} x_t \le k, \\ & x_t = \{0,1\}, \forall t. \end{aligned}$$

OSCC: Basic Setting

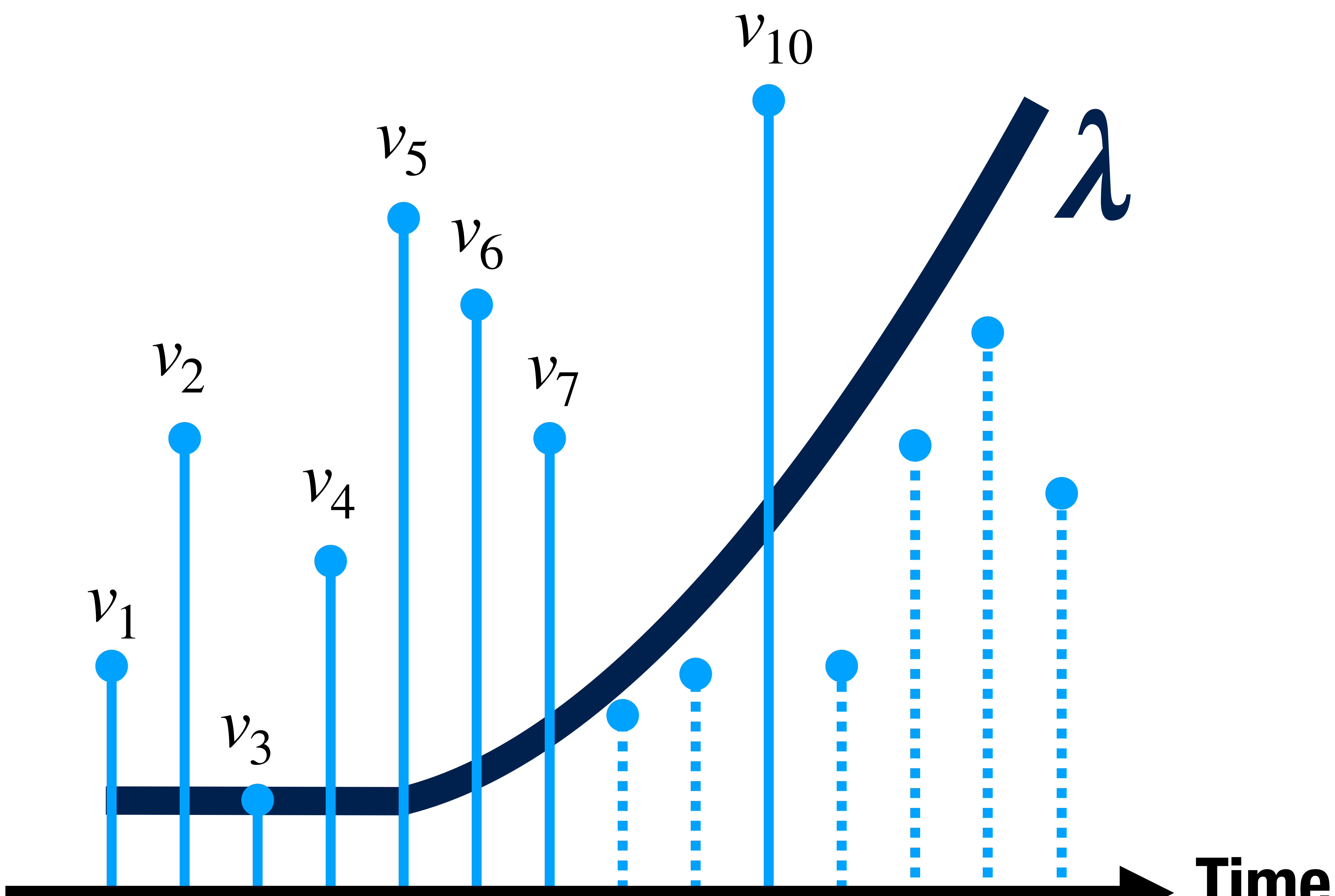


$$\alpha = \max_{\sigma \in \Omega_{\mathcal{S}}} \frac{\mathrm{Opt}(\sigma)}{\mathbb{E}[\mathrm{Alg}(\sigma)]}$$

$$\mathcal{S} = (k, f, v_{\min}, v_{\max})$$

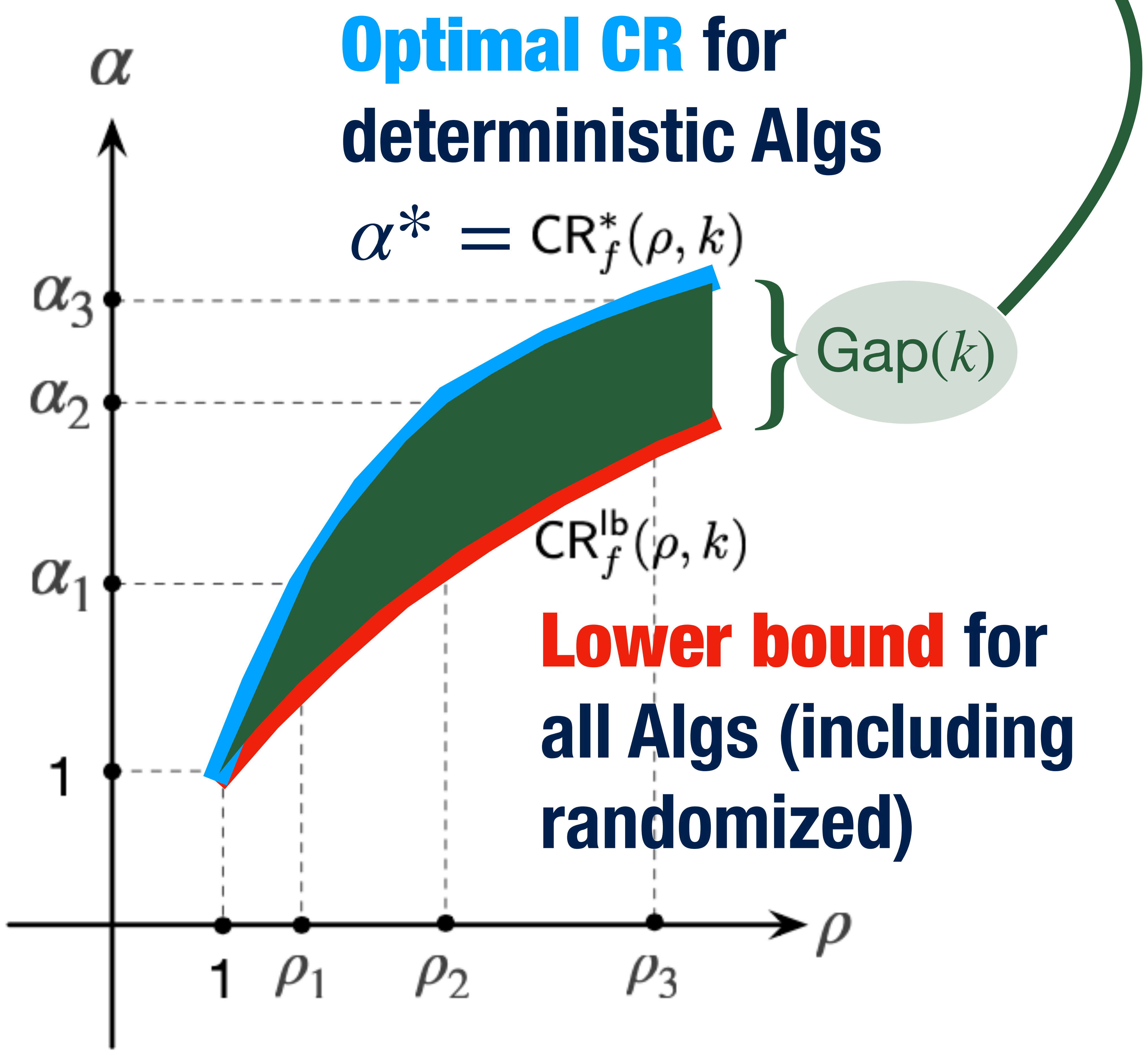
$$\begin{aligned}
\underset{\{x_t\}_{\forall t}}{\text{maximize}} \quad & \sum_{t=1}^{T} v_t x_t - f\left(\sum_{t=1}^{T} x_t\right) \\
\text{subject to} \quad & \sum_{t=1}^{T} x_t \le k, \\
& x_t = \{0,1\}, \forall t.
\end{aligned}$$

Threshold Policies with Tight Guarantees for OSCC

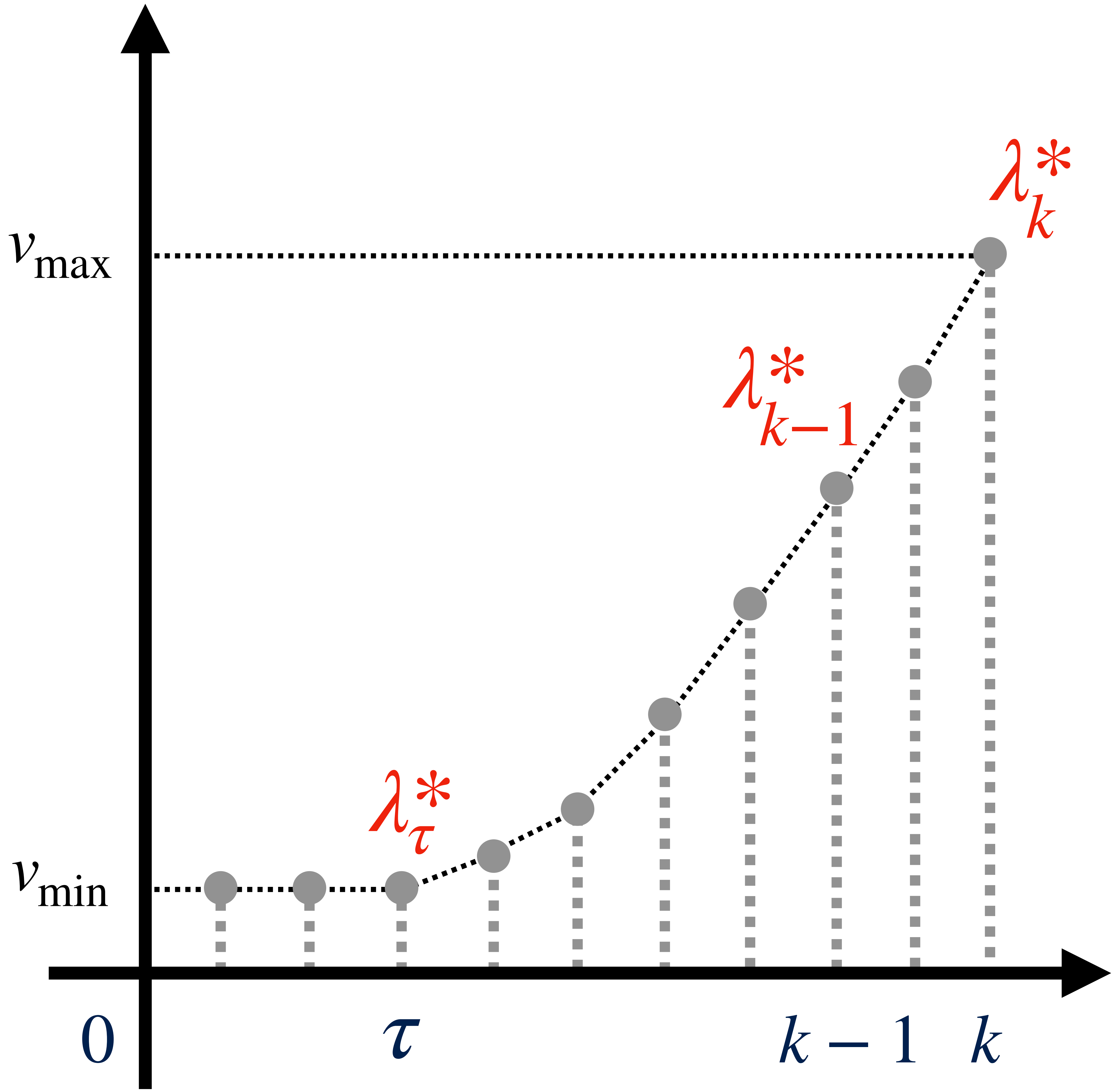


$$\rho = v_{\max}/v_{\min}$$

$$\lim_{k\to\infty} \mathsf{Gap}(k) = 0$$



Theorem 1: Optimal Threshold (Deterministic)



Theorem 1 (Optimal Threshold). *Given a setup $\mathcal{S} = \{f; p_{\min}, p_{\max}; k\}$, $\mathsf{TOS}_{\boldsymbol{\lambda}^*}$ achieves the optimal competitive ratio of all deterministic online algorithms, denoted by $CR_f^*(\rho, k)$, if and only if $\boldsymbol{\lambda}^* = \{\lambda_0^*, \lambda_1^*, \cdots, \lambda_\tau^*, \cdots, \lambda_k^*\}$ is an admission threshold such that*

- *The lower and upper limits: $\lambda_0^* = \lambda_1^* = \cdots = \lambda_\tau^* = p_{\min}$ and $\lambda_k^* = p_{\max}$.*
- *The turning point τ of the admission threshold is given by*

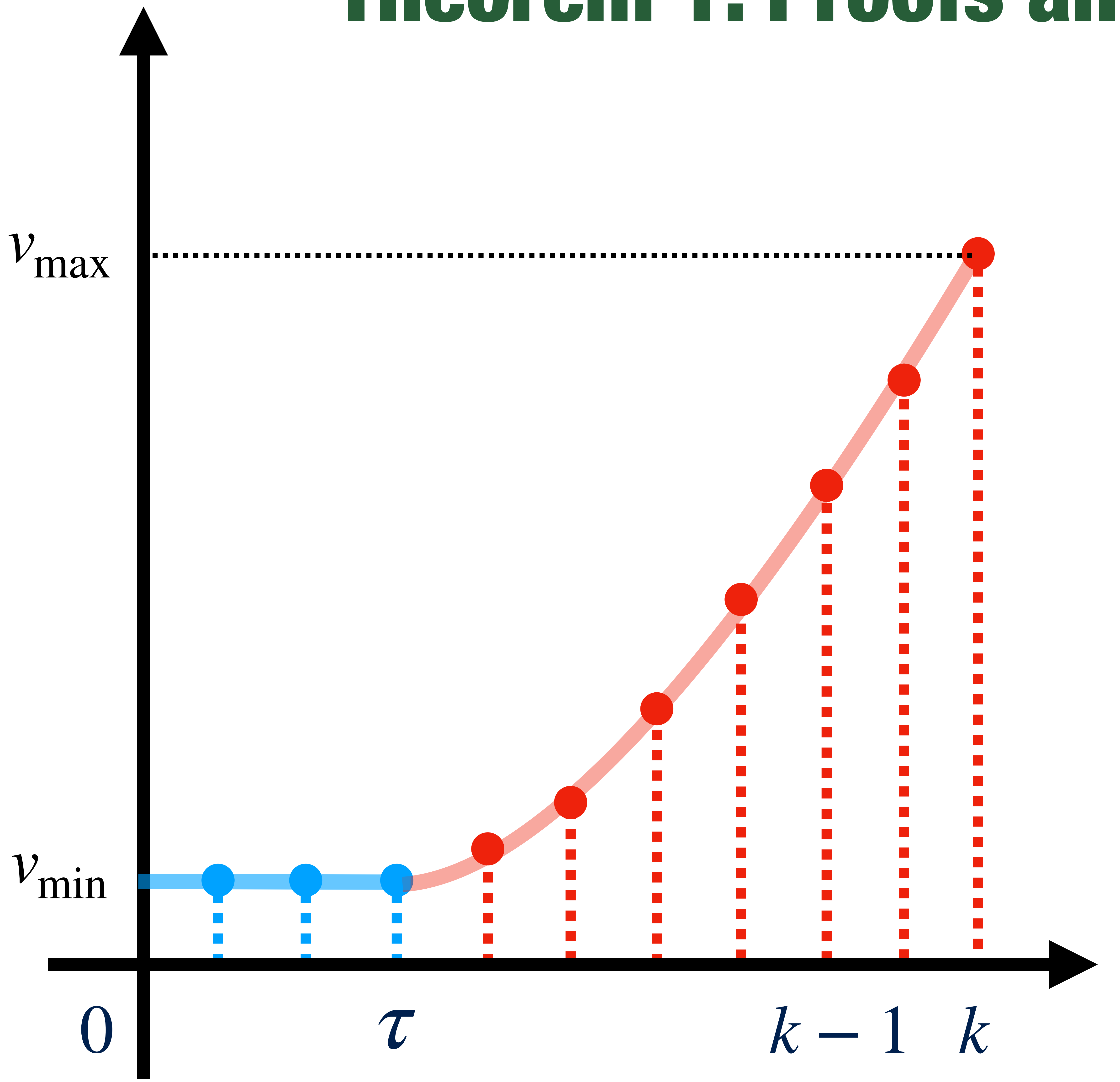
$$\tau = g^{\mathsf{inv}}\left(\frac{f^*(p_{\min})}{\mathsf{CR}_f^*(\rho, k)}\right) - 1. \tag{17}$$

- *The optimal competitive ratio $\mathsf{CR}_f^*(\rho, k)$ and $\{\lambda_{\tau+1}^*, \lambda_{\tau+2}^*, \cdots, \lambda_{k-1}^*, \lambda_k^*\}$ satisfy:*

$$\textbf{(SoSE):}\quad \mathsf{CR}_f^*(\rho, k) = \frac{f^*(\lambda_{\tau+1}^*)}{g(\tau+1)} = \frac{f^*(\lambda_{\tau+2}^*) - f^*(\lambda_{\tau+1}^*)}{\lambda_{\tau+1}^* - c_{\tau+2}} = \cdots = \frac{f^*(\lambda_k^*) - f^*(\lambda_{k-1}^*)}{\lambda_{k-1}^* - c_k}. \tag{18}$$

- **Existence** of many competitive TPs
- **Uniqueness** of optimal (deterministic) TP
- **Optimality** among all deterministic algs

Theorem 1: Proofs and Intuitions



$$f^*(p) \triangleq \max_{i\in\{0,1,\cdots,k\}} pi - f(i)$$

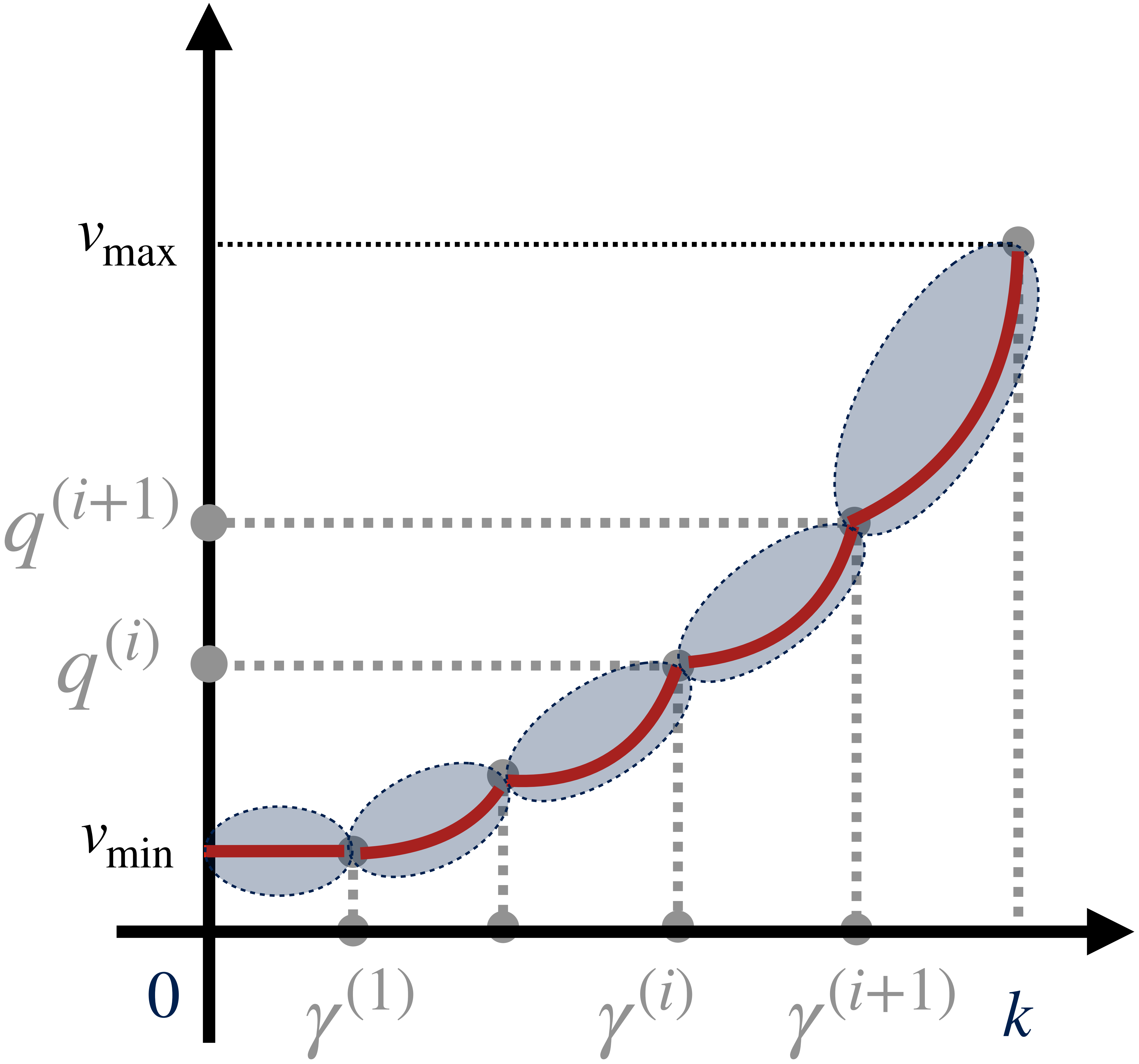
- Initial flat phase

$$v_{\min}(\tau+1) - f(\tau+1) \geq \frac{1}{\alpha} f^*(v_{\min})$$

- Strictly increasing phase

$$\Delta_{\text{primal}} \geq \frac{1}{\alpha} \cdot \Delta_{\text{dual}}$$

Theorem 2: Lower Bound



Theorem 2 (Lower Bound). *For any given setup $\mathcal{S} = \{f, p_{\min}, p_{\max}, k\}$, no online algorithms (possibly randomized) is $\left(\mathsf{CR}^{\mathsf{lb}}_f(\rho, k) - \epsilon\right)$-competitive for any $\epsilon > 0$, where $\mathsf{CR}^{\mathsf{lb}}_f(\rho, k)$ is given by*

$$\mathsf{CR}^{\mathsf{lb}}_f(\rho, k) = \frac{p_{\min}\underline{k} - f(\underline{k})}{p_{\min}\gamma^{(1)} - f(\gamma^{(1)})}, \tag{19}$$

where $\gamma^{(1)}$ is a real value within $(0, \underline{k}]$. Specifically, let us define the right-hand-side of Eq. (19) as $F(\gamma^{(1)})$ to indicate that it is an explicit function of $\gamma^{(1)} \in (0, \underline{k}]$. Together with $\{\gamma^{(\ell)}\}_{\ell=\{2,\cdots,\bar{k}-\underline{k}+2\}}$, they form a unique set of increasing positive real numbers (i.e., $0 < \gamma^{(1)} < \gamma^{(2)} < \cdots < \gamma^{(\bar{k}-\underline{k}+1)} < \gamma^{(\bar{k}-\underline{k}+2)} = \bar{k}$) that satisfy

$$\frac{q^{(\ell+1)}(\underline{k}+\ell-1)}{\exp\left(\frac{F(\gamma^{(1)})}{\underline{k}+\ell-1}\gamma^{(\ell+1)}\right)} - \frac{q^{(\ell)}(\underline{k}+\ell-1)}{\exp\left(\frac{F(\gamma^{(1)})}{\underline{k}+\ell-1}\gamma^{(\ell)}\right)} = \int_{\gamma^{(\ell)}}^{\gamma^{(\ell+1)}} \frac{F(\gamma^{(1)})f'(y)}{\exp\left(\frac{F(\gamma^{(1)})}{\underline{k}+\ell-1}y\right)}dy, \quad \forall \ell = [\bar{k} - \underline{k} + 1], \tag{20}$$

where $q^{(\ell)} = c_{\underline{k}+\ell-1}$ for $\ell = \{2, 3, \cdots, \bar{k} - \underline{k} + 1\}$, $q^{(1)} = p_{\min}$, and $q^{(\bar{k}-\underline{k}+2)} = p_{\max}$.

Piece-wise continuous threshold w/ fractional end points

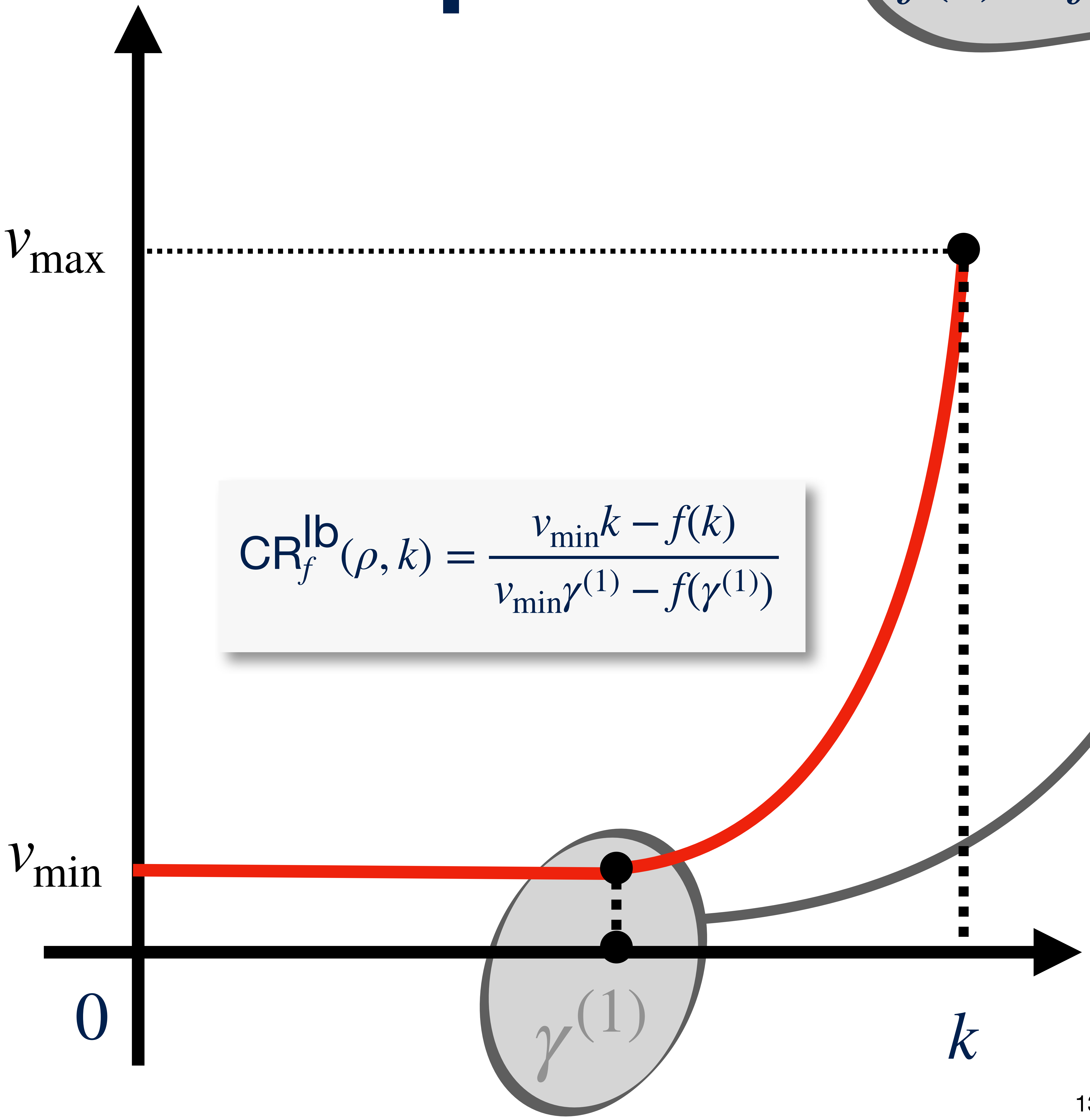
$$0 < \gamma^{(1)} < \gamma^{(2)} < \cdots < \gamma^{(\bar{k}-\underline{k}+1)} < \gamma^{(\bar{k}-\underline{k}+2)} = k,$$

$$v_{\min} = q^{(1)} < q^{(2)} < \cdots < q^{(\bar{k}-\underline{k}+1)} < q^{(\bar{k}-\underline{k}+2)} = v_{\max},$$

A special case $f(k) - f(k-1) < v_{\min}$

Even the highest (marginal) cost is still less than the lowest possible value

$$\frac{v_{\max}}{\exp(\alpha)} - \frac{v_{\min}}{\exp\left(\gamma^{(1)}\alpha/k\right)} = \frac{1}{k}\int_{\gamma^{(1)}}^{k} \frac{\alpha f'(y)}{\exp\left(y\alpha/k\right)} dy$$



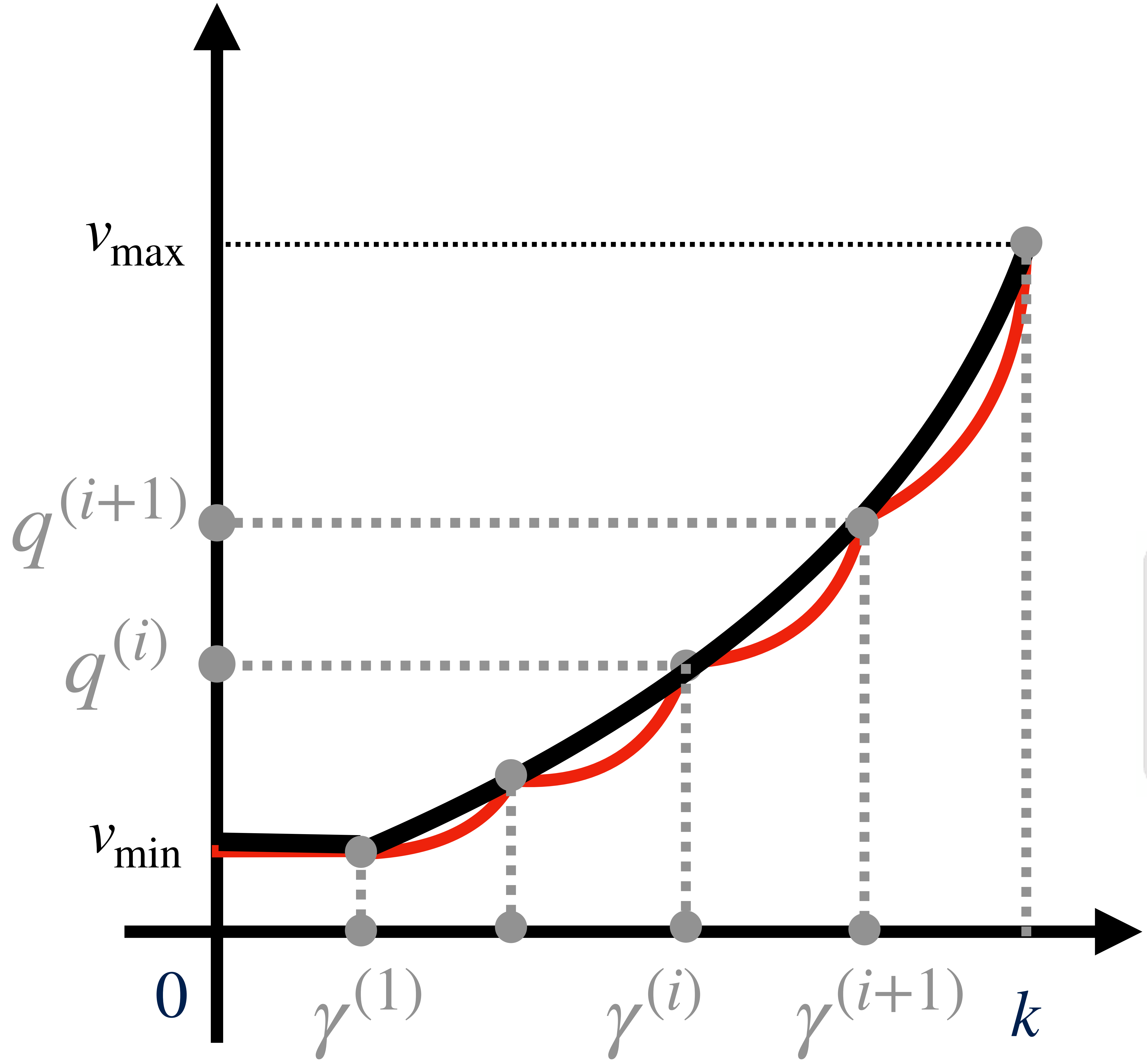
$$\mathrm{CR}_f^{\mathrm{lb}}(\rho, k) = \frac{v_{\min}k - f(k)}{v_{\min}\gamma^{(1)} - f(\gamma^{(1)})}$$

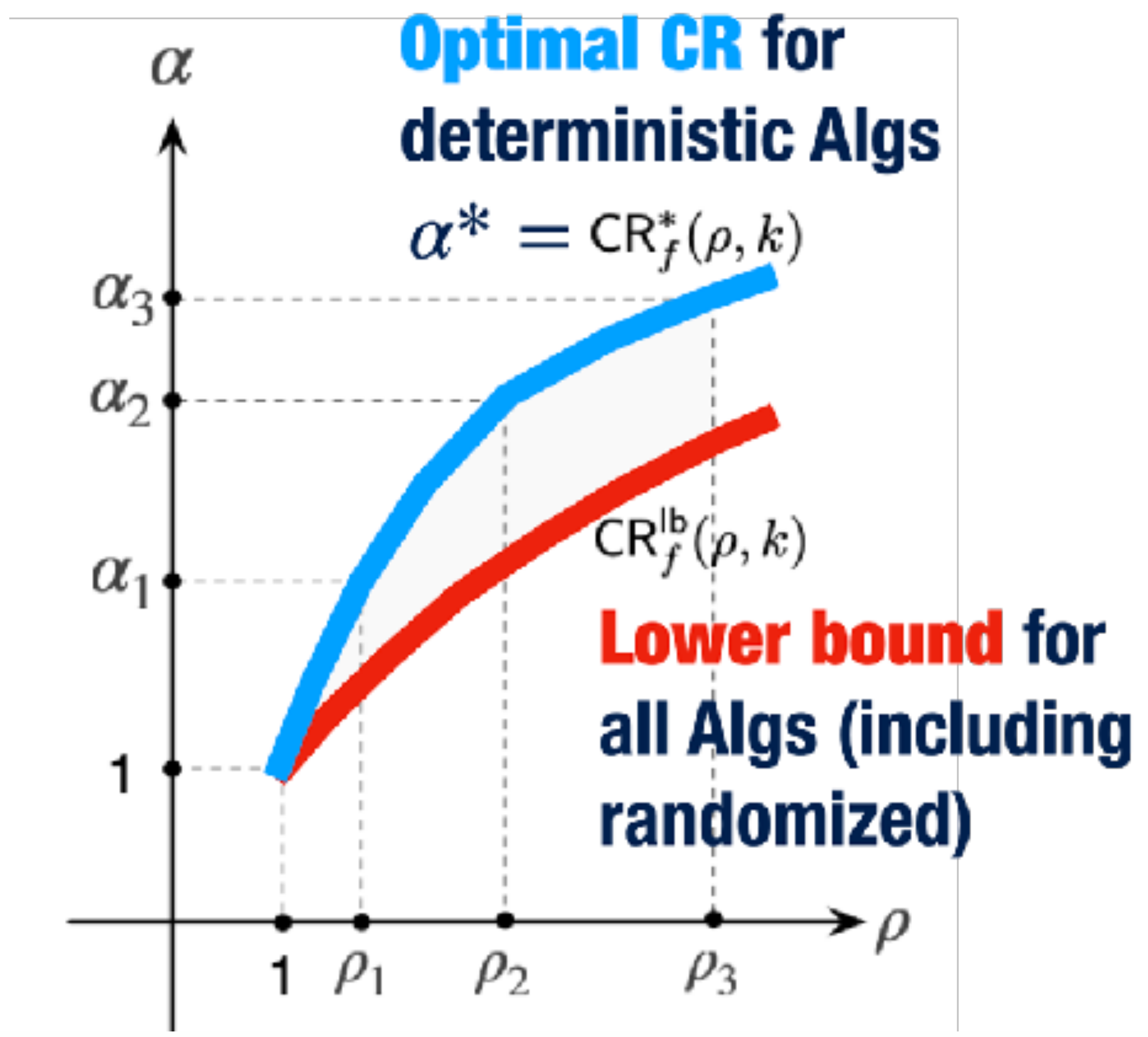
Online selection w/o costs: $f = 0$

$$\gamma^{(1)} = \frac{k}{1 + \ln \rho}$$

$$\mathrm{CR}_f^{\mathrm{lb}}(\rho, k) = 1 + \ln \rho$$

Theorem 3: Asymptotic Convergence



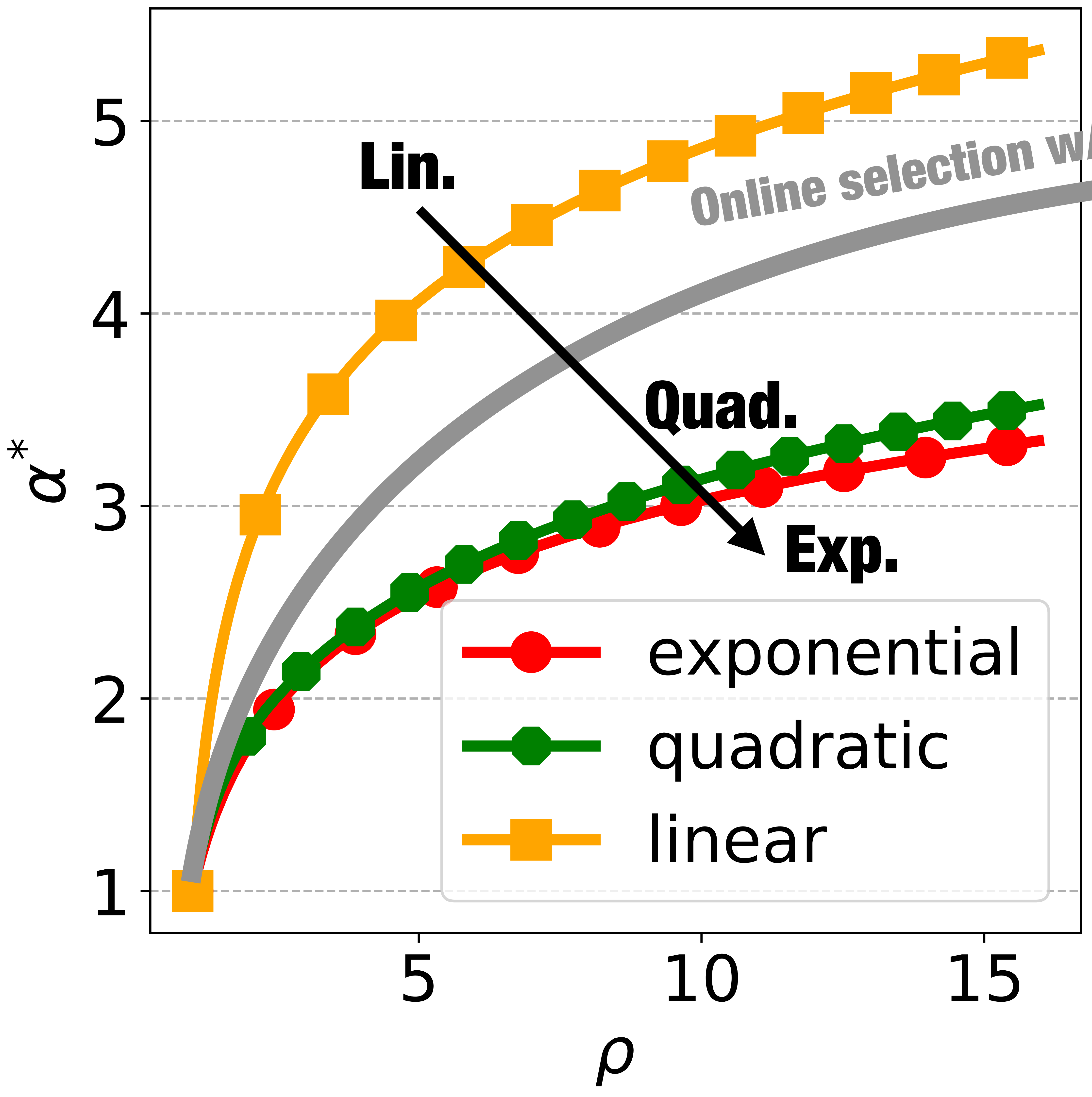


Theorem 3 (Asymptotic Lower Bound). *For any given setup $\mathcal{S} = \{f, p_{\min}, p_{\max}, k\}$, TOS_{λ^*} is asymptotically optimal among all online algorithms (including those with randomization) when $\bar{k}$ is sufficiently large, namely,*

$$\lim_{\bar{k}\to\infty} \mathsf{CR}_f^*(\rho, k) = \lim_{\bar{k}\to\infty} \mathsf{CR}_f^{\mathsf{lb}}(\rho, k) = \underline{\mathsf{CR}}_f(\rho), \tag{22}$$

where $\underline{\mathsf{CR}}_f(\rho)$ is the asymptotic lower bound that depends on f and ρ only.

A two-segment threshold is enough if k is sufficiently large!!



Online selection w/o costs: $f = 0$

$1 + \ln \rho$

Strongly convex

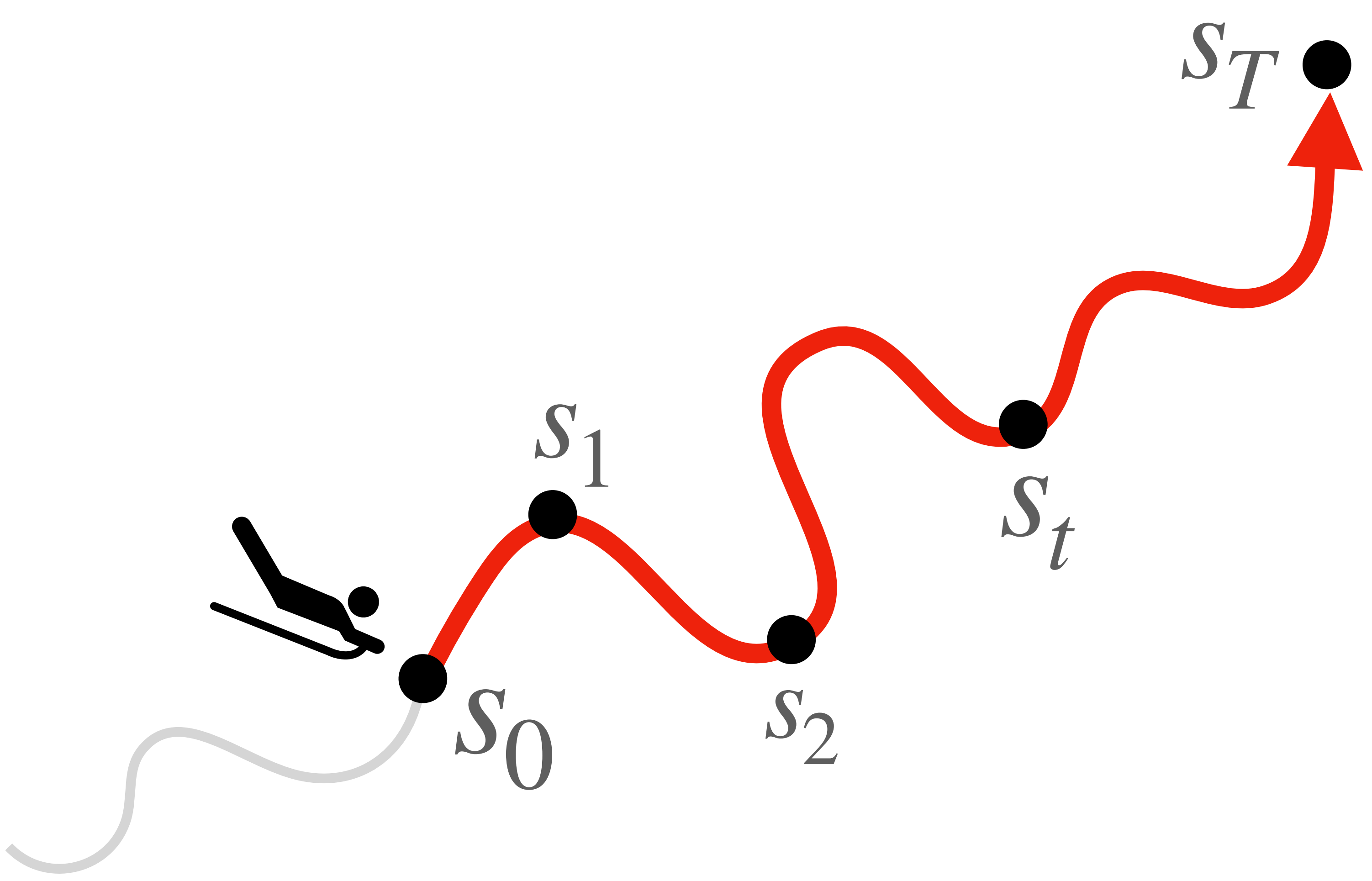
Higher value fluctuations $\Longrightarrow$

worse guarantees **(larger α^*)**

Faster cost growth $\Longrightarrow$

better guarantees **(smaller α^*)**

Generalization and Variations



$$\max_{x_t \in \mathbb{X}_t} \sum_t v_t(x_t) - f(s)$$

$$s = (s_0, s_1, s_2, \cdots, s_t, \cdots, s_T)$$

$$s_t \sim s_{t-1}, x_t$$

Action
- Convex
- Box
- Ramping
- Inventory
- Deadlines
- Binary

Value
- Linear
- Concave
- Strongly concave
- Separable
- Combinatorial

Cost
- Linear
- Quadratic
- Polynomial
- Convex
- Strongly convex
- Separable
- Non-convex

Dynamics
- Linear / Nonlinear
- Stationary / Non-stationary
- Too complex

Proceedings of Machine Learning Research vol 75:1–21, 2018 31st Annual Conference on Learning Theory

Smoothed Online Convex Optimization in High Dimensions via Online Balanced Descent

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COLT 2018

Regularized Online Allocation Problems: Fairness and Beyond

Santiago Balseiro [*1 2] **Haihao Lu** [*3 2] **Vahab Mirrokni** [2]

ICML 2021

Optimal Regularized Online Allocation by Adaptive Re-Solving

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(July 18, 2023)

Chasing Convex Bodies Optimally

Mark Sellke
Stanford University

Abstract

In the *chasing convex bodies* problem, an online player receives a request sequence of N convex sets $K_1, \dots, K_N$ contained in a normed space X of dimension d. The player starts at $x_0 = 0 \in X$, and at time n observes the set K_n and then moves to a new point $x_n \in K_n$, paying a cost $||x_n - x_{n-1}||$. The player aims to ensure the total cost exceeds the minimum possible total cost by at most a bounded factor α_d independent of N, despite x_n being chosen without knowledge of the future sets $K_{n+1}, \dots, K_N$. The best possible α_d is called the competitive ratio. Finiteness of the competitive ratio for convex body chasing was proved for $d = 2$ in [FL93] and conjectured for all d. [BLLS19] recently resolved this conjecture, proving an exponential $2^{O(d)}$ upper bound on the competitive ratio.

We give an improved algorithm achieving competitive ratio d in any normed space, which is *exactly* tight for ℓ^∞. In Euclidean space, our algorithm also achieves competitive ratio $O(\sqrt{d \log N})$, nearly matching a $\sqrt{d}$ lower bound when N is subexponential in d. Our approach extends that of [BKL+20] for *nested* convex bodies, which is based on the classical Steiner point of a convex body. We define the *functional* Steiner point of a convex function and apply it to the associated work function.

SODA 2020

Chasing Convex Bodies with Linear Competitive Ratio

C.J. Argue [1], Anupam Gupta[1], Guru Guruganesh[2], and Ziye Tang[1]

[1]Carnegie Mellon University
[2]Google Research

January 8, 2020

Abstract

We study the problem of chasing convex bodies online: given a sequence of convex bodies $K_t \subseteq \mathbb{R}^d$ the algorithm must respond with points $x_t \in K_t$ in an online fashion (i.e., x_t is chosen before K_{t+1} is revealed). The objective is to minimize the sum of distances between successive points in this sequence. Bubeck et al. (STOC 2019) gave a $2^{O(d)}$-competitive algorithm for this problem. We give an algorithm that is $O(\min(d, \sqrt{d \log T}))$-competitive for any sequence of length T.

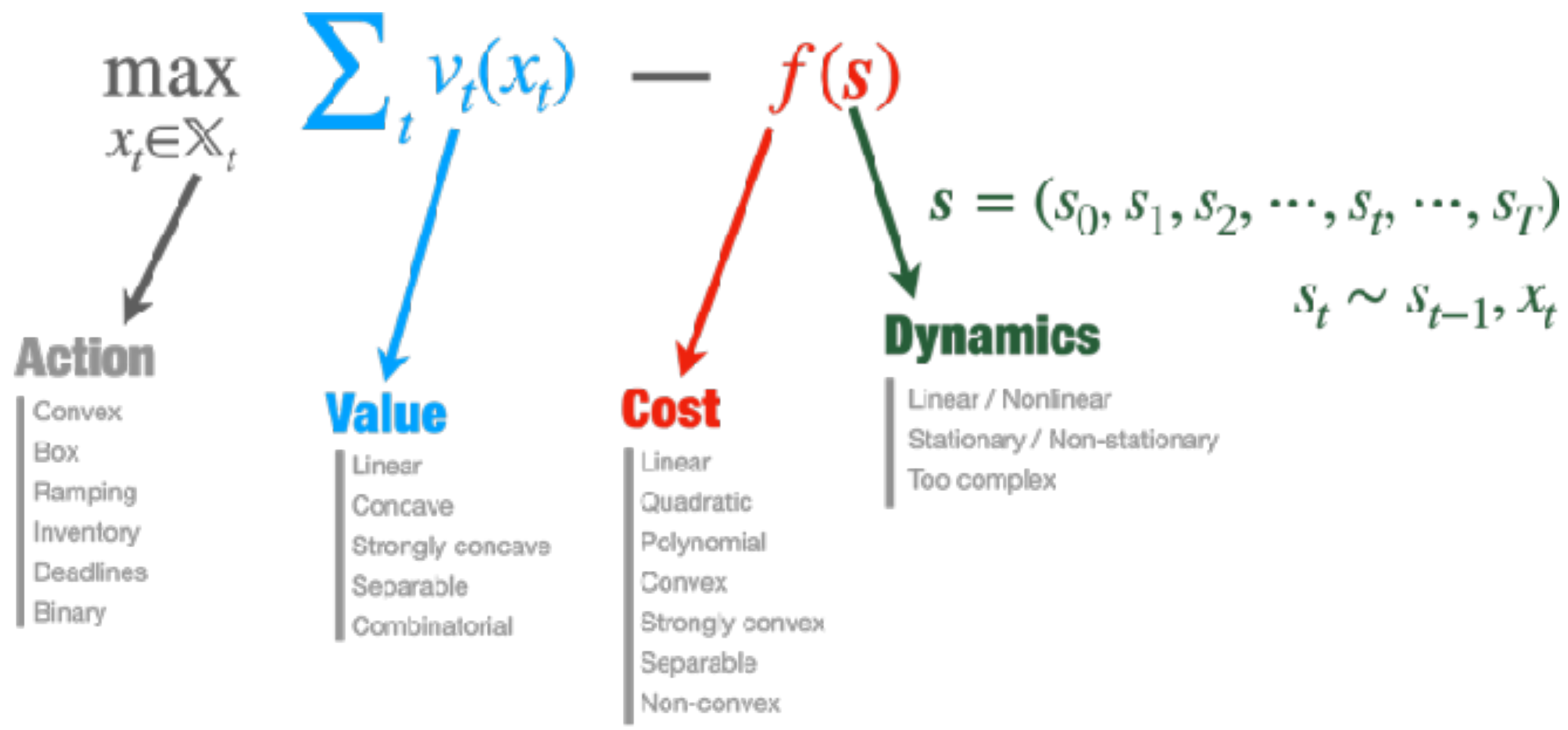
SODA 2020

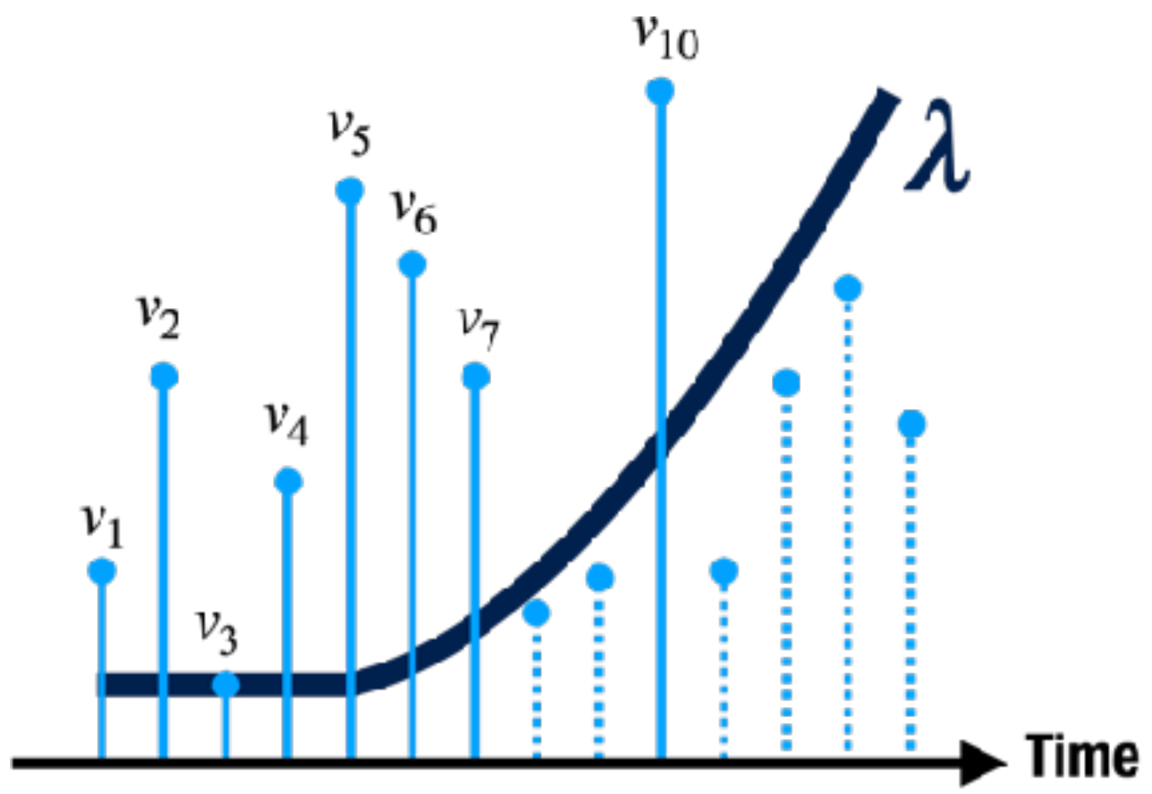
• • • **Many more**

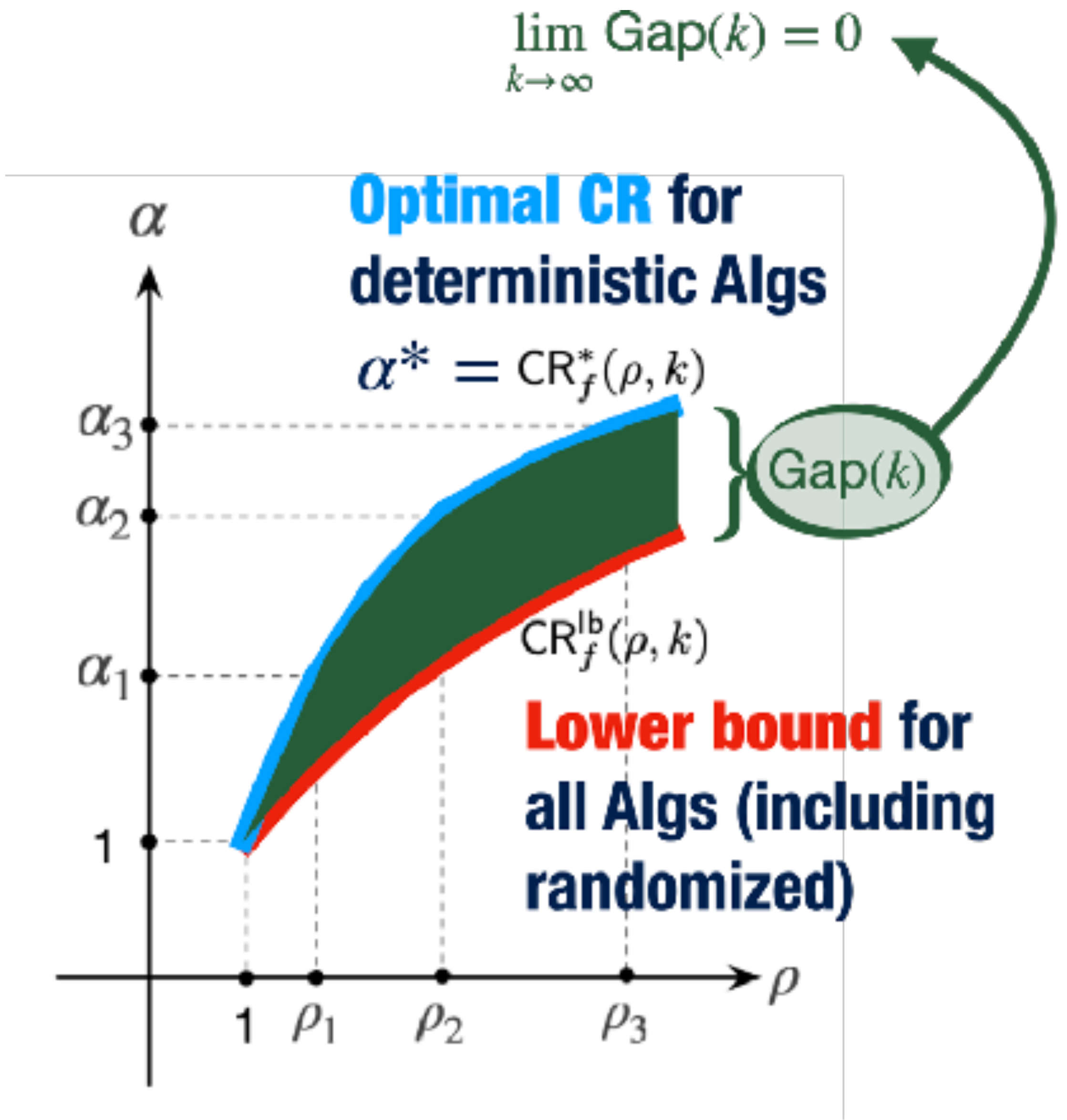
Concluding Remarks

- **Online selection** with **convex costs**
- **Simple** algorithms; **Strong** guarantees
- Variations; Open questions

- A set of T items (one at a time)
- Select a subset $S \subset T$ of items (with possible constraints)
- Maximize objective $v(S) - f(S)$







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